Semiclassical analysis of traversal time through Kac's solution of the telegrapher's equation

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A path-integral solution of the telegrapher's equation has been demonstrated to give a plausible description of traversal time, for motions either above or below the top of the barrier, in connection with microwave-simulation experiments [see Mugnai, Ranfagni, Ruggeri, and Agresti, Phys. Rev. Lett. **68**, 259 (1992)]. This Brief Report reports an extension of the analysis in order to compare the traversal, or delay, time results relative to a beat envelope signal with those as deduced from the distribution function of the randomized time and its analytical continuation in imaginary time.

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Since Kac's [1] pioneering work, it has been well known that the telegrapher's equation is equivalent to a stochastic motion where the jump rate is related to the dissipative parameter a in the following equation:

$$\frac{1}{v^2}\frac{\partial^2 F}{\partial t^2}+\frac{2a}{v^2}\frac{\partial F}{\partial t}=\frac{\partial^2 F}{\partial x^2}, \tag{1}$$

v being the propagation velocity in the x direction of the quantity $F(x,t)$. Kac's work has been recently reconsidered [2] and it was demonstrated that the solution of Eq. (1) can be expressed by the quadrature [3]

$$F(x,t)=\int_{-\infty}^{\infty}[\alpha\phi(x,r)+\beta\phi(x,-r)]\,g(t,r)\,dr\ , \tag{2}$$

where $\phi(x,r)$ is a solution of the wave equation without dissipation [Eq. (1) with $a=0$] and α and β are arbitrary mixing coefficients so that $\alpha+\beta=1$. The boundary conditions of Eq. (1) are $F(x,0)=\phi(x,0)$ and $(\partial F/\partial t)_{t=0}=(\alpha-\beta)(\partial\phi/\partial t)_{t=0}$. The two-variable function $g(t,r)$ is the density distribution of a "randomized time" which, according to a Laplace-transform analysis, can be expressed, for $-t<r\le t$, as

$$\begin{aligned}g(t,r)&=e^{-at}\delta(t-r)\\&+\frac{1}{2}ae^{-at}\Theta(t-|r|)[I_0(a(t^2-r^2)^{1/2})\\&+(t+r)(t^2-r^2)^{-1/2}I_1(a(t^2-r^2)^{1/2})]\ ,\end{aligned} \tag{3}$$

where $\delta(t)$ and $\Theta(t)$ are the Dirac function and the Heaviside step function, respectively, and I_0 and I_1 are modified Bessel functions. The properties of the function (3) are extensively discussed in Ref. [3]; here we recall just some of them and in particular the asymptotic behavior for $t\gg r$. Using the asymptotic expansions of the Bessel functions [4],

$$I_{0,1}(y)\underset{y\to\infty}{\sim}\frac{e^y}{\sqrt{2\pi y}}\ ,$$

and the approximate relation $(t+r)(t^2-r^2)^{-1/2}\approx e^{r/t}$, we have that Eq. (3) can be expressed, neglecting the δ contribution, as the sum of two Gaussians. Thus, we have

$$g(t,r)\approx\frac{1}{2}\sqrt{\frac{a}{2\pi t}}\left[\exp\left(-\frac{ar^2}{2t}\right)+\exp\left(-\frac{ar^2}{2t}+\frac{r}{t}\right)\right]\ , \tag{4}$$

where the first Gaussian is centered at $r=0$, the other one at $r=1/a$, with the tails cut at $r=\pm t$ (see Fig. 1). By means of Eq. (4) we account immediately for the fact that the asymptotic value $1/2a$ of the average $\bar{r}(t)$ is in agreement with the complete calculation, which exactly gives [3]

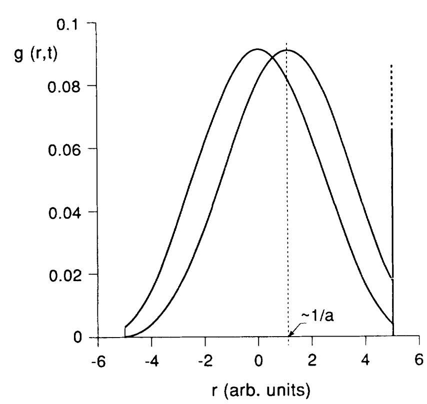

FIG. 1. The contributions relative to I_0 and I_1 [second and third terms in Eq. (3)] are computed as a function of r for $a=1$ and $t=5$. These can be approximated in the interval $-t\le r\le t$ as two nearly identical Gaussians normalized to 1/2, one centered at $r=0$ and the other one at $r\simeq 1/a$; see Eq. (4). The first term in Eq. (3) is an attenuated δ function, centered at $r=t$, which for $at\gg 1$ becomes negligible.

$$\bar{r}(t) = \int_{-\infty}^{\infty} r g(t,r)\, dr = \frac{1}{2a}\left(1 - e^{-2at}\right) . \tag{5}$$

Note that in the integration of Eq. (5) the contributing terms of the distribution $g(t,r)$ are, for parity reasons, only those in the following expression ($0 \le r \le t$):

$$f(r,t) = e^{-at}\left\{\delta(t-r) + a\frac{r}{\sqrt{t^2-r^2}} \times I_1(a\left(t^2-r^2\right)^{1/2})\Theta(t-r)\right\} .$$

This function, by identifying r with r_1, is identical to the density distribution of the first passage time [Eq. (6) in Ref. [5] for $r_0 = 0$]. As a consequence, we have that the average $\bar{r}_1$

$$\bar{r}_1(t) = \int_0^{\infty} r_1\, f(r_1,t)\, dr_1 = \frac{1}{2a}\left(1 - e^{-2at}\right)$$

is coincident with the average $\bar{r}(t)$ of Eq. (5). We wish to recall that the same result of Eq. (5) was originally obtained by Kac [1] evaluating the first moment $\mu_1(t) \equiv \bar{r}$ of the randomized time by the average

$$\mu_1(t) = \left\langle \int_0^t (-1)^{N(\tau)}\, d\tau \right\rangle = \int_0^t \left\langle (-1)^{N(\tau)} \right\rangle d\tau , \tag{6}$$

where $N(\tau)$ is a randomized variable with Poisson distribution of intensity a. By computing the average in the last term of relation (6) as a sum of all probabilities, Kac simply obtained

$$\left\langle (-1)^{N(\tau)} \right\rangle = e^{-2a\tau} , \tag{7}$$

which substituted into Eq. (6) just gives Eq. (5). Note that if we considered the *amplitude* of probability (indeed of probability) we would obtain a special average (a sort of transition element)

$$\tilde{r}(t) = \int_0^t \left|\left\langle (-1)^{N(\tau)} \right\rangle\right|^{1/2} d\tau = \frac{1}{a}\left(1 - e^{-at}\right) \tag{8}$$

which is formally identical to the one of Eq. (5) by substituting $2a \to a$.

The average time $\bar{r}$ (or $\tilde{r}$) has to be interpreted as the fictitious time it would take a particle to reach the average distance $L = v\bar{r}$ if it was always moving with velocity v without reversal. Accordingly, Eq. (5) [and Eq. (8)] clearly accounts for the fact that dissipation continuously reduces the effective speed of the motion tending the average distance to the saturation value $v/2a$. By multiplying by v and inverting Eq. (5) we have that the average *true* time required to reach the distance $L \equiv v\bar{r}$ is given by

$$t = -\frac{1}{2a}\ln\left(1 - 2a\frac{L}{v}\right) \tag{9}$$

[the same result is obtained by Eq. (8) with the substitution $2a \to a$]. For $a \to 0$, or $(L/v) \to 0$, Eq. (9) correctly gives the classical result $t = L/v$, while for the saturation value $L = v/2a$, t tends to infinity.

Let us now compare this result with the one obtained by considering a simple signal function. In the case of a single sinusoidal progressive wave of the type $\phi(x,t) = \sin(x - vt)$ [6], Eq. (2) for $\alpha = 1$ and $\beta = 0$ turns out to be

$$F(x,t) = e^{-at}\left(\sin x \cos w_1 t - \frac{v}{w_1}\cos x \sin w_1 t + \frac{a}{w_1}\sin x \sin w_1 t\right) , \tag{10}$$

where $w_1 = \sqrt{v^2 - a^2}$ is an effective velocity, lower than the velocity v in the absence of dissipation. By identifying the v velocity with that of a beat envelope (beat or group velocity), formed by a superposition of two waves of sligtly different frequencies, it was possible to give a reasonable description of experimental results of delay time measured in guided propagation of electromagnetic waves in the region of the cutoff frequency [7]. For $v > a$, the delay (or traversal) time can be simply obtained as the time t_1 taken for a node of the envelope function [$F(x,t) = 0$] to travel a distance $x = L$ from the relation [8,9]

$$\tan(w_1 t_1) = \frac{w_1 \tan L}{v - a\tan L} . \tag{11}$$

The behavior of t_1 vs the dissipative parameter a is represented in Fig. 2 and compared with the curve as given by Eq. (9), taking a or $2a$ as an independent variable. We note that the agreement is reasonable especially when the

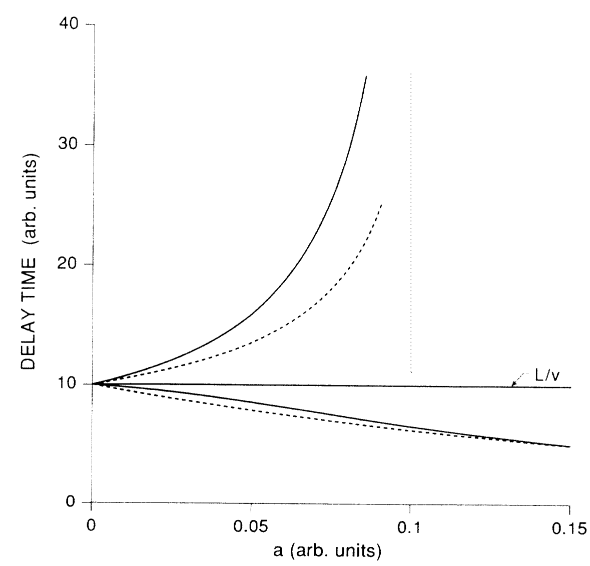

FIG. 2. Delay or traversal time relative to a barrier of unitary length ($L = 1$) as a function of the dissipative parameter a. The upper curves refer to a classically allowed motion with initial velocity $v = 0.1$, the lower ones refer to a classically forbidden motion with an initial imaginary velocity $v = 0.1i$. Continuous lines refer (upper) to Eq. (11) and (lower) to Eq. (12). Dashed lines refer (upper) to Eq. (9) and (lower) to Eq. (14), taking $2a$ as an independent variable. The horizontal line represents the delay in the absence of dissipation.

independent variable is $2a$. So we can conclude that the predictions of delay time vs dissipation are described well enough by both models [Eqs. (9) and (11)] in the allowed classical region.

Now we wish to investigate the behavior of delay (or traversal) time for a classically forbidden region or quantum-tunneling regime. According to an already tested procedure [7,9], when the velocity v becomes imaginary ($v^2 < 0$) we consider the analytic continuation of Eq. (10) in imaginary time ($t \to -it$) and imaginary effective velocity $w_1 \to iw_3 = i\sqrt{|v|^2 + a^2}$, since $v \to iv$. In this way the amplitude $F(x,t)$ turns out to be a complex quantity and the delay time can be obtained by equating to zero the derivative of $|F|^2$ with respect to time, and obtaining therefore the following relation [8,9]:

$$\tan(2w_3t_3) = \frac{w_3}{v}\tan(2L)\,. \tag{12}$$

The curve of t_3 vs the dissipative parameter a, as deduced from Eq. (12), is also reported in Fig. 2. This curve shows, in opposition to the classical behavior, a characteristic *decrease* of the delay time with increasing dissipation. This behavior, which has been demonstrated to work in practical cases [7,9], seems to be peculiar of the tunneling processes where (contrarily to the classically allowed region where dissipation tends to reduce the motion speed) the effective imaginary velocity is *increased* by dissipation.

We ask now if it is possible to compare the result as deduced from Eq. (12) with something like Eq. (9), deduced from Eq. (3), in the tunneling case.

The distribution $g(t,r)$ as well as the average effective randomized time $\bar{r}(t)$, Eq. (5), are direct consequences of the stochastic model assumed for describing the effect of losses. These assumptions lead, as previously anticipated, to the intuitive result of a slowing down of the motion. In the tunneling case we have just the opposite behavior, but it is not easy to imagine the corresponding motion model. The opposite behavior of the delay vs dissipation suggests to us to consider as trial function for $\bar{r}(t)$ the *inverse* function of Eq. (5), namely (after exchanging $\bar{r}$ and t)

$$\bar{r} = -\frac{1}{2a}\ln(1-2at) \tag{13}$$

and, for the special average $\tilde{r}$, the inverse function of Eq. (8). In this way the traversal (or tunneling) time would be given—by multiplying by v, taking again $L \equiv v\bar{r}$, and inverting Eq. (13)—by

$$t = \frac{1}{2a}\left(1 - e^{-2aL/v}\right) \tag{14}$$

[the same result is obtained by Eq. (8) with the substitution $2a \to a$]. The corresponding curve is also reported in Fig. 2 and we note that (especially taking $2a$ as an independent variable) the agreement with the curve deduced from Eq. (12) is rather good, better than in the allowed-motion case [comparing the curves relative to Eqs. (9) and (11) in Fig. 2].

This result, if not fortuitous, suggests that the crude *ansatz* of Eq. (13) should have some more convincing explanation. First, we note that Eq. (14) is nothing but Eq. (5) with the substitution $\bar{r} \to t$ and vice versa $t \to \bar{r} \equiv L/v$. This means that, while for classically allowed motions the effective space is $L = v\bar{r}$ and the true time is t, in the tunneling case the effective space is $L = vt$ and the true time is $\bar{r}$, that is, the roles of t and $\bar{r}$ are exactly *exchanged* when passing from classical to tunneling motions. This interchange of the time variables t and r is compatible with the structure of the distribution function $g(t,r)$ when we consider an analytic continuation g_{AC} of the type

$$g_{AC}(t,r) \propto g(it,ir)\,. \tag{15}$$

Looking at Eq. (3) we see that the argument of the modified Bessel functions exactly becomes $a(r^2-t^2)^{1/2}$ (this implies a trivial exchange of r and t) so that after a suitable change in the argument of the Θ function we have that the asymptotic expression of $g(it,ir)$ is given, for $r \gg t$, by

$$g(it,ir) \approx \mathcal{N}\,\frac{a}{2}\,\frac{e^{-iat}e^{ar}}{\sqrt{2\pi ar}}\left[\exp\left(-\frac{at^2}{2r}\right) + i\exp\left(-\frac{at^2}{2r}+\frac{t}{r}\right)\right], \tag{16}$$

where $\mathcal{N}$ is a suitable normalization factor. This expression allows us to obtain for the asymptotic absolute value of the average time $\bar{t}$ the following result:

$$\bar{t}(r) = \left|\int_{-\infty}^{\infty} it\, g(it,ir)\, d(it)\right| \underset{r\to\infty}{\sim} \frac{1}{2a}\,, \tag{17}$$

in agreement with the asymptotic value of Eq. (14). A more detailed analysis could allow us to obtain the complete expression (14). Nevertheless, the essential issue of this kind of operation is evidenced also by the above simplified treatment.

What clearly emerges is the inverted role of dissipation that in tunneling processes acts as an "accelerator" of the motion. This fact, which finds a natural interpretation when dissipation is treated by a phenomenological approach [10], is difficult to explain in connection to a real stochastic motion—even in imaginary time—which should result from the superposition of "undisturbed" normal processes and "disturbed" ones, which in tunneling behave as accelerated processes. The implications of this shortening of the tunneling time are ultimately connected to superluminal transport properties which for the tunneling have been theoretically predicted [11] and, in special situations [12], experimentally verified [13–15].

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